

Kelly bet sizing¹

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¹ as discussed in p. 74 of

Aaron Brown. *Red-Blooded Risk*. Wiley Finance, 2011

This document derives Kelly's rule for optimal bet sizing.

HERE'S THE GAME. You have a \$10,000 stake. You face a series of 60 wins and 40 losses. Their order is chosen by me. You choose each bet's size. After you tell me the size, I get to spend, or not, one of my 40 kills—if I have any left. Your win is my loss, and vice-versa. After 100 rounds, we're done.

If you always bet the same fixed percentage of your stake, then² I have no strategic role to play in determining your overall takeaway. I can't affect your outcome if you play that way.

² since multiplication is commutative

The Math

I'll denote by a the aggressiveness of your bets. If you win a bet then $\text{stake}_{t+1} = \text{stake}_t \cdot (1 + a)$ and if you lose a bet then $\text{stake}_{t+1} = \text{stake}_t \cdot (1 - a)$. A win, then a loss, turns out the same as a loss, then a win:

$$\begin{aligned}\text{stake}_{t+2} &= (1 + a) \cdot (1 - a) \cdot \text{stake}_t \\ &= (1 - a) \cdot (1 + a) \cdot \text{stake}_t.\end{aligned}$$

³ assuming, to remove my strategic input, that it's the same fraction a every time

In these terms, Kelly's question is: what is the optimal³ aggressiveness a^* to **maximise your final takeaway**⁴?

⁴ i.e., $\arg \max_{\{a\}} \text{stake}_{100} \stackrel{\text{def}}{=} a^*$

The Answer

Twenty percent is the perfect amount to bet. Betting a higher or a lower fraction means doing worse. ... While there are people who dislike the Kelly criterion for various reasons, no intelligent person disputes this aspect of the result.

—Brown, p. 76

Solving for optimal aggressiveness

This can be solved with calculus ¹⁰¹. Since the order doesn't matter and there's a fixed number of wins and losses, we can write the following formula for your total winnings at 60/40 odds:

$$\$1000 \cdot (1 + a)^{60} \cdot (1 - a)^{40} \quad (\text{take})$$

.

To find the optimal aggressiveness, set the derivative of equation (take), with respect to a , equal to zero.⁵ The symbolic derivative of

$$(1 + a)^{60} \cdot (1 - a)^{40}$$

with respect to a is:

$$60 \cdot (1 + a)^{59} \cdot (1 - a)^{40} - 40 \cdot (1 + a)^{60} \cdot (1 - a)^{39} \quad (\text{D})$$

. Setting (D) = 0 tells me a property that is true of the optimal a^* .

Moving things around, that property can be restated as:

$$60 \cdot \frac{(1 - a^*)^{40}}{(1 - a^*)^{39}} = 40 \cdot \frac{(1 + a^*)^{60}}{(1 + a^*)^{59}}$$

which reduces to the much nicer

$$60 \cdot (1 - a^*) = 40 \cdot (1 + a^*).$$

Solving then for the optimal aggressiveness a^* :

$$60 - 60a^* = 40 + 40a^* \quad (1)$$

$$20 = 100a^* \quad (2)$$

which is what Brown gets: 20% = $\frac{1}{5}$ for the optimal a^* against these odds.

Brown's conclusion

The "Ed Thorp takeaway" is that risk management does not mean taking no risks. You can't sit on your hands. Bill Gross says to "avoid portfolio mush", which is similar & related.

Contrast this to "lazy CAPM" style thinking. When you have an edge (like a 60/40 edge) you need to exploit it. Betting less than 20% of your stake against 60/40 odds is⁶ suboptimal. Less aggressive betting does not move you along an optimal frontier; it moves you *off* the optimal frontier.

References

Aaron Brown. *Red-Blooded Risk*. Wiley Finance, 2011.

⁵ (There are further conditions to make sure this works, which I'm leaving out.)

The product rule says $D(f \cdot g) = D(f) \cdot g + f \cdot D(g)$.

Thanks, Artemy!

⁶ In this model.